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LETTER TO THE EDITOR

A general integral of the axially symmetric stationary Einstein equations

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Abstract. The Ernst function of an axially symmetric stationary asymptotically flat space-time involving an arbitrary harmonic function and an arbitrary number of constants is presented and discussed.

As is well known, the metric of a stationary axially symmetric Einstein field can be calculated by simple mathematical operations from a solution $f(z, \bar{z})$, $W(z, \bar{z})$ ($W = \bar{W}$) of the Ernst equations

$$(Wf_{,z})_{,\bar{z}} + (Wf_{,\bar{z}})_{,z} = 4W(f+\bar{f})^{-1}f_{,z}f_{,\bar{z}} \qquad W_{,z,\bar{z}} = 0 \tag{1}$$

where a bar denotes complex conjugation. For $W =$ constant these equations are equivalent to a sinh-Gordon equation and are not connected with gravitational fields. Nevertheless the algebraic method for solving the sinh-(sine-)Gordon equation by repeated Bäcklund transformations has stimulated the search for a similar procedure applicable to the Ernst equations. Such a method was found (Neugebauer 1979) and transformed into a recursion formula (Neugebauer 1980) which, working as a non-linear creation operator, generates from a given solution f_0, W_0 of (1) new solutions with any number of constants. The only analytic work to be done is to solve the total Riccati equations (Neugebauer 1979)

$$\mathrm{d}\gamma = W_0^{-1}(\gamma-1)(\gamma W_{0,z}\,\mathrm{d}z + W_{0,\bar{z}}\,\mathrm{d}\bar{z}) \tag{2}$$

$$\begin{aligned}\mathrm{d}\alpha = {} & (f_0+\bar{f}_0)^{-1}([\alpha-\gamma^{1/2}]\bar{f}_{0,z} + [\alpha^2\gamma^{1/2}-\alpha]f_{0,z})\,\mathrm{d}z \\ & + (f_0+\bar{f}_0)^{-1}([\alpha-\gamma^{-1/2}]\bar{f}_{0,\bar{z}} + [\alpha^2\gamma^{-1/2}-\alpha]f_{0,\bar{z}})\,\mathrm{d}\bar{z}\end{aligned} \tag{3}$$

which are completely integrable if f_0, W_0 are given solutions of (1). The functions $\alpha(z, \bar{z})$ and $\gamma(z, \bar{z})$ have to satisfy the conditions

$$\bar{\alpha} = \alpha^{-1} \qquad \bar{\gamma} = \gamma^{-1}. \tag{4}$$

From α and γ (or quantities derived from α and γ) the new solutions of (1) can be calculated algebraically. In this Letter we present an (asymptotically flat) solution involving an arbitrary number of integration constants, which is a result of an arbitrary number of recursion steps (Bäcklund transformations). In this regard it should be noted that Belinsky and Zakharov (1979), Cosgrove (1979), Herlt (1978), Hoenselaers *et al* (1979) and Kinnersley and Chitre (1978) have also used systematic methods for solving the Ernst equations (or equations equivalent to the Ernst equations) and have obtained many interesting results. Our solution can be shown to be given by

$$f=f_0\frac{\det(c_{ik}[\sqrt{\gamma_k}]^i)}{\det(b_{ik}[\sqrt{\gamma_k}]^i)} \qquad W=W_0 \tag{5}$$

where c_{ik}, b_{ik} are $(2N+1)\times(2N+1)$ matrices (N a positive integer) defined in the following way:

$$\begin{aligned} &c_{(l+2)m}=c_{lm} \qquad c_{0m}=1 \qquad c_{1m}=\alpha_m \qquad \text{for } m=0,1,2,\ldots,2N \\ &b_{(l+2)m}=b_{lm} \qquad b_{0m}=1 \qquad b_{1m}=\begin{cases} 1 & \text{for } m=0 \\ \alpha_m & \text{for } m=1,2,\ldots,2N. \end{cases} \end{aligned} \tag{6}$$

Furthermore we have to choose

$$\gamma_0=1, \qquad \alpha_0=\bar{f}_0/f_0. \tag{7}$$

The functions γ_k and α_k are solutions of the Riccati equations (2) and (3) with different integration constants indicated by the index k.

The general solution of $W_{0,z,\bar{z}}=0$ can always be transformed into the form $W_0=\frac{1}{2}(z+\bar{z})$. Then $\rho=\mathrm{Re}\, z$, $\zeta=\mathrm{Im}\, z$ become cylindrical coordinates and the functions γ_k are given by

$$\gamma_0=1 \qquad \gamma_k=\frac{a_k-\mathrm{i}\bar{z}}{a_k+\mathrm{i}z} \qquad (k=1,2,\ldots,2N) \qquad a_k=\bar{a}_k \tag{8}$$

where $\{a_k\}(k=1,2,\ldots,2N)$ is a countable set of real integration constants. The static Weyl class is an important particular case, in which the functions α_k can be calculated explicitly. In addition to (5) let us give a more detailed form to the Ernst function f:

$$\frac{f}{f_0}=\frac{\begin{vmatrix} 1 & 1 & 1 & \ldots & 1 \\ \alpha_0 & \alpha_1\sqrt{\gamma_1} & \alpha_2\sqrt{\gamma_2} & \ldots & \alpha_i\sqrt{\gamma_i} \\ 1 & (\sqrt{\gamma_1})^2 & (\sqrt{\gamma_2})^2 & \ldots & (\sqrt{\gamma_i})^2 \\ \alpha_0 & \alpha_1(\sqrt{\gamma_1})^3 & \alpha_2(\sqrt{\gamma_2})^3 & \ldots & \alpha_i(\sqrt{\gamma_i})^3 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & (\sqrt{\gamma_1})^i & (\sqrt{\gamma_2})^i & \ldots & (\sqrt{\gamma_i})^i \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \alpha_1\sqrt{\gamma_1} & \alpha_2\sqrt{\gamma_2} & \ldots & \alpha_i\sqrt{\gamma_i} \\ 1 & (\sqrt{\gamma_1})^2 & (\sqrt{\gamma_2})^2 & \ldots & (\sqrt{\gamma_i})^2 \\ 1 & \alpha_1(\sqrt{\gamma_1})^3 & \alpha_2(\sqrt{\gamma_2})^3 & \ldots & \alpha_i(\sqrt{\gamma_i})^3 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & (\sqrt{\gamma_1})^i & (\sqrt{\gamma_2})^i & \ldots & (\sqrt{\gamma_i})^i \end{vmatrix}} \tag{9}$$

where $i=2N$.

The special case $N=2$ of the solution (5) (or (9)) has been discussed by Kramer and Neugebauer (1979). The following interpretations generalise the results obtained for $N=2$.

If f_0 corresponds to an asymptotically flat space–time, the functions α_k can be chosen in such a way that the space–time connected with f is also asymptotically flat. In this case the solution f describes a nonlinear superposition of the given solution f_0 and N Kerr–NUT solutions. It should be noted that NUT-like singularities (angular momentum monopoles) of f can be removed by means of an Ehlers transformation.

Now let us specify the functions f_0, W_0. We first consider the flat space–time solution $f_0 = 1$, $W_0 = \frac{1}{2}(z + \bar{z})$. The functions γ_k are given by (8). By using (3) it can readily be verified that the functions α_k are constants. A particular choice for these constants leads to

$$\alpha_0 = -1 \qquad \alpha_k = (-1)^{k+1} \exp[\mathrm{i}(-1)^k \varphi_k] \qquad \varphi_{2k} = \varphi_{2k-1} (k = 1, 2, \ldots, N) \tag{10}$$

and this describes the superposition of N Kerr particles, where the differences $(a_{2k} - a_{2k-1})$ $(k = 1, 2, \ldots, N)$ are proportional to the masses m_k $(k = 1, 2, \ldots, N)$ of the individual particles and the real constants φ_{2k} $(k = 1, 2, \ldots, N)$ are the rotation parameters for these particles. In order to obtain the superposition of N Schwarzschild particles we put $\varphi_k = 0 (k = 1, 2, \ldots, 2N)$. In this case the determinants are of the Vandermonde type and can therefore be factorised. If the given solution belongs to the static Weyl class $f_0 = \bar{f}_0 = \mathrm{e}^{2u}$, $W_0 = \frac{1}{2}(z + \bar{z})$, the coordinate dependence of the functions α_k is found from (4), (7) and (3) to be

$$\alpha_k = \frac{l_k + \mathrm{i}\,\mathrm{e}^{2\phi_k}}{l_k - \mathrm{i}\,\mathrm{e}^{2\phi_k}} \qquad l_k = \bar{l}_k = \text{constant} \qquad (k = 0, 1, \ldots, 2N) \tag{11}$$

$(\phi_0 = u)$, where $\{l_k\}$ is a countable set of real integration constants. The function $\phi(z, \bar{z})$ is a line integral over derivatives of the axisymmetric harmonic function $U = \frac{1}{2} \ln f_0$,

$$\mathrm{d}\phi_k = (\gamma_k)^{1/2} U_{,z}\, \mathrm{d}z + (\gamma_k)^{-1/2} U_{,\bar{z}}\, \mathrm{d}\bar{z}. \tag{12}$$

It does not depend on the path of integration. By a particular choice of the constants l_k it is always possible to satisfy the requirement that for $\phi_k = 0$ the functions α_k $(k = 0, 1, \ldots, 2N)$ in (11) should agree with the flat space α_k in (10). This solution corresponds to an asymptotically flat space–time involving a harmonic function and N mass and angular momentum parameters.

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